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MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

# Relaxation Methods Applied to Engineering Problems IX. High-Speed Flow of Compressible Fluid through a Two-Dimensional Nozzle

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# RELAXATION METHODS APPLIED TO ENGINEERING PROBLEMS

## IX. HIGH-SPEED FLOW OF COMPRESSIBLE FLUID THROUGH A TWO-DIMENSIONAL NOZZLE

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Methods developed in Parts III and V of this series are here extended to the more difficult problem of compressible fluid moving at high speed through a convergent-divergent nozzle. Solutions of sufficient accuracy for practical purposes can be obtained for a nozzle of any specified shape, provided that the velocity of the fluid nowhere exceeds the local speed of sound. Otherwise the computed velocities fail to converge—a result similar to what was obtained by Taylor and Sharman using an electrical tank.

The reason of this failure is discussed, and an alternative method (*not* in itself entailing the 'relaxation' technique) is proposed to meet the difficulty. In a subsequent paper this will be applied to determine the supersonic regime.

### Introduction and Summary

1. The equations which govern the motion of compressible fluid (e.g. the flow of steam through a nozzle) present great difficulties even when the fluid is assumed to be inviscid and as such to 'slip' freely over a solid boundary. Osborne Reynolds's well-known treatment (1886) in effect assumed the velocity to be uniformly distributed over each cross-section, therefore dealt with a problem rendered one-dimensional by assumption. In two dimensions very few exact solutions have been discovered, and these relate to cases (e.g. line-vortices, and radial or spiral flow to or from a sink or source) which are not realizable in practice. The basic difficulty, of course, is the inapplicability of the principle of superposition.

Rayleigh (1916) attacked the two-dimensional problem by a method of successive approximation, but even in the case of flow past a circular cylinder he was not able to proceed very far. Bryan (1918) suggested a semi-graphical method which has not in fact led to results. Taylor & Sharman (1928) utilized an electrical analogue to obtain solutions by experiment: their method is one of 'cut-and-try', and was not successful when the local speed of sound was exceeded at any point.

This being the state of theory, a numerical method which is general will have value even though it be incapable of giving very exact results. Here we present a relaxational treatment of nozzles which as we believe has amply sufficient accuracy for practical purposes. Like the electrical tank it fails when the local speed of sound is attained in some part of the field, and in our concluding section we propose an alternative method to meet this difficulty.

Although most of the material now presented has been in existence for some two years, we had not intended publication until a complete solution (covering the 'supersonic' regime) could be presented. Now, however, circumstances are altered in that one of us (J. R. G.) will be leaving this country shortly for Australia; and on that account (since the new method was developed only in the last few weeks) it has seemed desirable to record this partial treatment now, leaving the supersonic regime for discussion later. It has still to be established that the alternative method does in fact converge under supersonic conditions.

## I. General Theory

*The governing equations*

2. Assuming the motion to be irrotational, we can express the component velocities $u$ and $v$ as derivatives of a velocity potential $\phi$. Moreover, the equation of continuity

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0 \tag{1}$$

(which states that there can be no steady accumulation of mass at any point) permits the introduction of a 'mass-flow function' $\psi$. Accordingly we may write

$$\frac{1}{\rho}\frac{\partial \psi}{\partial y}=u=-\frac{\partial \phi}{\partial x}, \quad -\frac{1}{\rho}\frac{\partial \psi}{\partial x}=v=-\frac{\partial \phi}{\partial y}, \tag{2}$$

and then, eliminating $\psi$ and $\phi$ in turn from (2), we have

$$\frac{\partial}{\partial x}\left(\rho\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\rho\frac{\partial \phi}{\partial y}\right)=0 \tag{3}$$

and

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial y}\right)=0 \tag{4}$$

as alternative forms of the governing equation.

3. The difficulty of our problem lies in the circumstance that $\rho$ is additionally related with $u$ and $v$ in virtue of its dependence on the local pressure. If we postulate that there is no transfer of energy between contiguous elements of the fluid,* and if the fluid is assumed to start from rest in a reservoir where its state is known, then its total energy will have the same value at every point; so if $p$, $\rho$ and $q$ denote the pressure, density and velocity of the fluid, and if body forces are inoperative or negligible, then

$$\int_0 \frac{dp}{\rho}+\tfrac{1}{2}q^2=0 \tag{5}$$

when the lower limit of integration relates to the starting conditions, so that $q_0=0$. Moreover $p$ and $\rho$ will conform with some known law of adiabatic expansion, e.g. with the relation

$$p/\rho^{\gamma}=\text{const.}=p_0/\rho_0^{\gamma} \tag{6}$$

in the case of a perfect gas, $\gamma$ denoting the ratio of the specific heats at constant pressure and at constant volume. Consequently by eliminating $p$ we can relate the density and velocity at any point in the fluid field.

* Already, in postulating irrotational motion, we have excluded the possibility of transfer due to friction and heat conduction will be negligible when the speed of flow is high.

As obtained from (5) and (6) the relation can be written in the form

$$q^2 = \frac{2}{\gamma-1}(a_0^2 - a^2), \tag{7}$$

$a$ standing for the velocity of sound at the point in question, so that

$$a^2 = \gamma p/\rho = \gamma p_0 \rho^{\gamma-1}/\rho_0^\gamma \quad \text{according to (6).} \tag{8}$$

On our assumptions, (7) and (8) will hold exactly in respect of a perfect gas, and with sufficient accuracy ($\gamma$ having the value 1·4) in respect of air. For steam, (6) and therefore (7) will hold approximately when $\gamma$ is replaced by an appropriate constant of the order of 1·3 (Ewing 1926, § 134). Alternatively, (7) and (8) may be replaced by an empirical relation between $\rho$ and $q^2$.

4. Thus for any fluid we can relate $\rho$ and $q^2$, therefore $\rho$ and $\rho^2 q^2$; i.e. the form of $F$ is known in the expression

$$\left.\begin{aligned} \chi^2 \text{ (say)} = \frac{1}{\rho} &= F(\rho^2 q^2) \\ &= F\left\{\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right\} \end{aligned}\right\} \tag{9}$$

by (2), since $q^2 = u^2 + v^2$. Equation (4) can now be written as

$$\frac{\partial}{\partial x}\left(\chi^2 \frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\chi^2 \frac{\partial\psi}{\partial y}\right) = 0,$$

and this in turn (multiplied throughout by $1/\chi$) as

$$\nabla^2(\chi\psi) - \psi\nabla^2\chi = 0, \tag{10}$$

$\nabla^2$ denoting the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

We shall take (9) and (10) as the governing equations of our problem, combined with boundary conditions which (for a symmetrical nozzle) may be written as

$$\left.\begin{aligned} \psi &= 0 \text{ on the centre-line,} \\ \psi &= \text{const.} = M \text{ (say), on the nozzle wall.} \end{aligned}\right\} \tag{11}$$

Then, according to (2)

$$2M \text{ measures the total mass flow through the nozzle.} \tag{12}$$

*Elimination of 'dimensional' factors*

5. Computation being essentially a *numerical* process, at the outset it is desirable to eliminate 'dimensional' factors from the governing equations.

First, the relation between $\rho$ and $q$ can be expressed as a relation between $\rho/\rho_0$ and $q/a_0$, $a_0$ being the velocity of sound which corresponds with the initial state of the fluid (viz. of rest in the reservoir). According to (7) and (8) we have

$$\left(\frac{q}{a_0}\right)^2 = \frac{2}{\gamma-1}\left(1 - \frac{a^2}{a_0^2}\right) = \frac{2}{\gamma-1}\left\{1 - \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\right\}, \tag{i}$$

and for steam a corresponding relation can be deduced from empirical data. Consequently (9) can be replaced by a relation between the purely numerical quantities $\rho/\rho_0$ and $\rho q/\rho_0 a_0$, viz.

$$\rho_0/\rho = F(\rho^2 q^2/\rho_0^2 a_0^2). \qquad \text{(ii)}$$

$F$ has, of course, a slightly different form in (ii) as compared with (9).

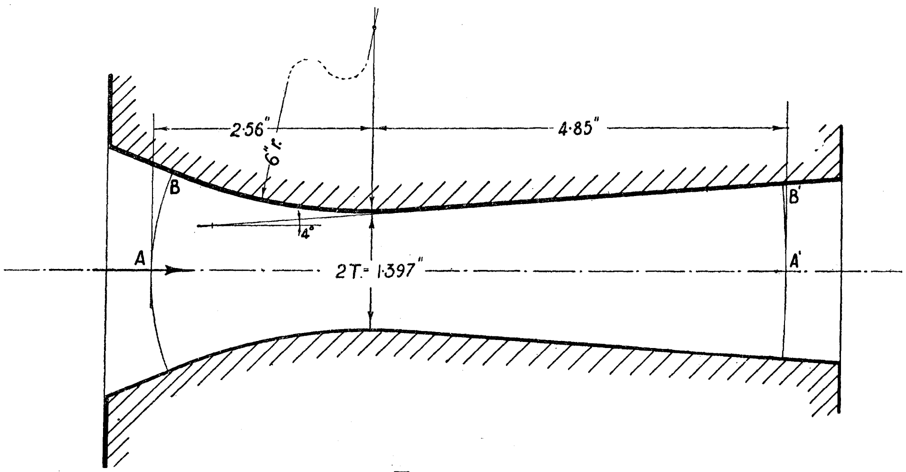


FIGURE 1

$$\left.\begin{array}{l}\text{Now writing} \qquad x = Tx', \quad y = Ty', \quad \rho = \rho_0\rho', \quad \psi = M\psi' \\ \text{and} \qquad \chi'^2 \text{ for } 1/\rho' = \rho_0\chi^2, \quad q' \text{ for } q/a_0, \\ \text{where } 2T \text{ (cf. figure 1) denotes the throat width of the nozzle, and} \\ \quad M \text{ has the significance stated in (12), so that} \\ \quad M/T \text{ measures the average mass flow per unit area of throat,}\end{array}\right\} \qquad (13)$$

we have from (2) and (13)

$$\rho^2 q^2 = \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = \frac{M^2}{T^2}\left\{\left(\frac{\partial\psi'}{\partial x'}\right)^2 + \left(\frac{\partial\psi'}{\partial y'}\right)^2\right\},$$

consequently (ii) may be written as

$$\chi'^2 = \frac{1}{\rho'} = F(\rho'^2 q'^2) = F\left[\mu^2\left\{\left(\frac{\partial\psi'}{\partial x'}\right)^2 + \left(\frac{\partial\psi'}{\partial y'}\right)^2\right\}\right], \qquad \text{(iii)}$$

where $\mu = M/\rho_0 a_0 T$, the average mass flow per unit throat area

*expressed as a fraction of* $\rho_0 a_0$. (14)

The form of (10) is conserved: i.e. on substituting from (13) we have

$$\left.\begin{array}{c}\nabla'^2(\chi'\psi') - \psi'\nabla'^2\chi' = 0, \\ \nabla'^2 \text{ denoting the operator } \partial^2/\partial x'^2 + \partial^2/\partial y'^2.\end{array}\right\} \qquad \text{(iv)}$$

Finally, on substituting from (13) in the boundary conditions (11) we have

$$\left.\begin{array}{rl}\psi' &= 0 \text{ on the centre line} \\ &= 1 \text{ on the nozzle wall.}\end{array}\right\} \qquad (15)$$

*Conformal transformation of the fluid field*

6. The problem for relaxation methods is to determine $\psi'$ with sufficient accuracy by computation, and for this we require its values at a large number of points in the 'field' between the two curved boundaries. The obvious procedure would be to employ a rectangular net conforming with the Cartesian coordinates $x'$, $y'$; but we can simplify the numerical computations by utilizing the device of conformal transformation, effected by the methods of Part V (Gandy & Southwell 1940).

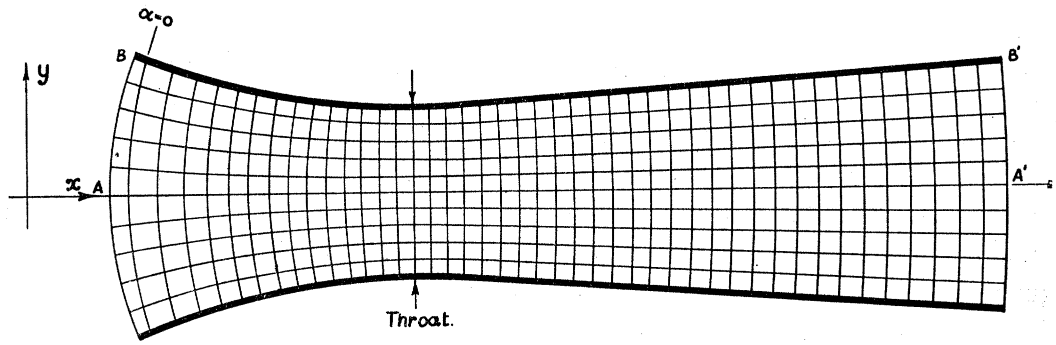


FIGURE 2*a*

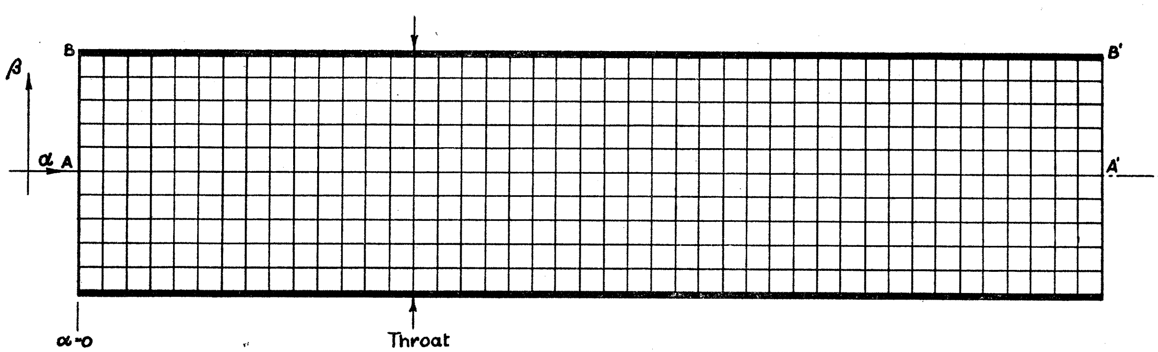


FIGURE 2*b*

Thereby we obtain the curvilinear net shown (diagrammatically) in figure 2*a*. It is composed of intersecting contours of two conjugate plane-harmonic functions $\alpha$ and $\beta$, of which $\beta$ is defined by the requirements that it vanishes on the centre line of the (symmetrical) nozzle, has a constant value $\beta_1$ (say) on the nozzle wall, and increases from 0 to that value at uniform rates along two circular arcs $AB$, $A'B'$ which cut the nozzle wall orthogonally at points $B$, $B'$ well upstream and well downstream of the 'throat'. The introduction of $AB$, $A'B'$ is obligatory because the field of computation must be limited: their shapes and positions are arbitrary, provided that the assumed variation of $\beta$ is such that contours of $\beta$ are stream-lines for an *incompressible* fluid passing right through the nozzle. Being dependent on the conditions at entry and at exit, these stream-lines can only be guessed. Our choice of circular arcs for $AB$, $A'B'$ and of uniform gradients for $\beta$ presumes that in the convergent and divergent parts of the nozzle the flow of the incompressible fluid is effectively radial and uniform, as in symmetrical flow to and from a point.

7. Thus specified, $\beta$ is calculable and contours of $\beta$ will be members of a family which includes the centre line and the wall of the nozzle. The conjugate function $\alpha$ can be deduced (apart from a nugatory constant of integration) by the methods developed in Part V, and its contours will cut the $\beta$-contours orthogonally. None of them (in general) will exactly coincide with $AB$, $A'B'$, because along those curves $\beta$ will not accord exactly with the stream function of an incompressible fluid passing right through the nozzle; but in fact the discrepancy is small and such that no sensible inaccuracy need be suspected in the region of the 'throat'.

Any convenient value $\beta_1$ may be attached to $\beta$ at the nozzle wall, and any convenient number of contours may be mapped: in our net (figure 7*b*) the contour values of $\beta$ increase from 0 to $10^5$ by increments of $2\times10^4$, and the same increments separate contour values of $\alpha$. A one-to-one correspondence obtains between points on the curvilinear net and points in the $\alpha$-$\beta$ plane; points on the centre line of the nozzle (where $\beta=0$) correspond with the axis of $\alpha$, and points on the nozzle wall with the horizontal line ($\beta=10^5$). The curvilinear 'field' of figure 2*a* is transformed into the ruled rectangle of figure 2*b*, and any physical quantity determined as a function of $\alpha$ and $\beta$ in figure 2*b* can also be plotted as a function of $x'$ and $y'$ in figure 2*a*. Consequently *velocities, etc. can be computed on a net of square mesh having no 'unequal stars'* (Part III, §§ 23–4), and this makes for considerable simplification.

8. Let $h$ stand for the modulus of transformation, so that

$$h^2=\left|\frac{d(\alpha+i\beta)}{d(x'+iy')}\right|^2=\left(\frac{\partial\alpha}{\partial x'}\right)^2+\left(\frac{\partial\alpha}{\partial y'}\right)^2=\left(\frac{\partial\beta}{\partial x'}\right)^2+\left(\frac{\partial\beta}{\partial y'}\right)^2. \tag{16}$$

Then in (iii) and (iv) of § 5 we may substitute

$$\left.\begin{aligned} &h^2\left\{\left(\frac{\partial\psi'}{\partial\alpha}\right)^2+\left(\frac{\partial\psi'}{\partial\beta}\right)^2\right\}\quad\text{for}\quad\left(\frac{\partial\psi'}{\partial x'}\right)^2+\left(\frac{\partial\psi'}{\partial y'}\right)^2\\ \text{and}\qquad &h^2\nabla^2_{\alpha,\beta}\quad\text{for}\quad\nabla'^2\end{aligned}\right\} \tag{17}$$

($\nabla^2_{\alpha,\beta}$ denoting the operator $\partial^2/\partial\alpha^2+\partial^2/\partial\beta^2$)

to obtain

$$\chi'^2=F\left[\mu^2h^2\left\{\left(\frac{\partial\psi'}{\partial\alpha}\right)^2+\left(\frac{\partial\psi'}{\partial\beta}\right)^2\right\}\right] \tag{18}$$

and

$$\nabla^2_{\alpha,\beta}(\chi'\psi')-\psi'\nabla^2_{\alpha,\beta}\chi'=0 \tag{19}$$

as governing equations of the transformed problem.

According to (15) the boundary conditions are

$$\left.\begin{aligned}\psi'&=0\quad\text{when}\quad\beta=0,\\ &=1\quad\text{when}\quad\beta=\beta_1,\end{aligned}\right\} \tag{20}$$

$\beta_1$ being the value so denoted in § 6. To make the problem definite we have only (*a*) to specify the shape of the nozzle and the value of the mass-flow constant $\mu$ as defined in (14), and (*b*) to define the form of the function $F$ in (18).

*The computational problem*

9. *Hereafter we shall suppress the dashes which have been attached to $\chi$ and $\psi$*, so that in what follows

$$\left.\begin{array}{l}\chi \text{ stands for } \rho_0/\rho;\ M\psi \text{ is the mass-flow function } (\S\,2).\\ \text{We shall, moreover, write } \nabla^2\text{, simply, for } \quad \nabla^2_{\alpha,\beta} \equiv \dfrac{\partial^2}{\partial\alpha^2}+\dfrac{\partial^2}{\partial\beta^2}.\end{array}\right\} \tag{21}$$

On this understanding, in 'non-dimensional' Cartesian co-ordinates $\alpha, \beta$, and on a rectangular net bounded by the straight lines $\beta = 0$, $\beta = 10^5$, our problem is to satisfy the equation

$$\nabla^2(\chi\psi) - \psi\nabla^2\chi = 0 \tag{19 A}$$

at every point, together with the relation

$$\chi^2 = F\left[\mu^2 h^2\left\{\left(\frac{\partial\psi}{\partial\alpha}\right)^2+\left(\frac{\partial\psi}{\partial\beta}\right)^2\right\}\right], \tag{18 A}$$

in which

$$\left.\begin{array}{l}h^2 \text{ has a known distribution (derived by conformal transformation, §§ 6–8),}\\ \mu = M/\rho_0 a_0 T \text{ is a specified ('mass-flow') parameter,}\end{array}\right\} \quad (14)\ bis$$

and the form of the function $F$ is known for the specified fluid. The quantity

$$\mu^2 h^2\left\{\left(\frac{\partial\psi}{\partial\alpha}\right)^2+\left(\frac{\partial\psi}{\partial\beta}\right)^2\right\} = \rho'^2 q'^2 = \rho^2 q^2/\rho_0^2 a_0^2, \tag{22}$$

where

$\rho$ denotes the local density,

$q$ denotes the local velocity,

$\rho_0$ denotes the density of the fluid at starting (from rest),

$a_0$ denotes the speed of sound in the fluid when at rest with density $\rho_0$, pressure $p_0$.

*Assumed state of fluid at starting. Relation of density to velocity*

10. In this paper we treat the case of air started from rest in a reservoir where its temperature is 15° C and its pressure 100 lb./sq.in. (absolute). Then for pound-foot-second units ($R = 96g$)

$$\frac{p_0}{\rho_0} = 96g \times 288 \quad \text{and} \quad p_0 = 100g \times 144;$$

therefore

$$\frac{1}{\rho_0} = \frac{96\times 288}{100\times 144} = 1{\cdot}92,$$

and giving to $\gamma$ for air the value 1·4 we deduce from (8) that

$$a_0^2 = \gamma p_0/\rho_0 = 1400g \times 144 \times 1{\cdot}92 = 12{,}463{,}718 \text{ ft.-sec. units} \quad (g = 32{\cdot}2).$$

Equation (i), § 5, becomes

$$q'^2 = \left(\frac{q}{a_0}\right)^2 = \frac{2}{\gamma-1}\left\{1-\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\right\} = 5(1-\rho'^{0\cdot4}),$$

consequently $$\rho'^2 q'^2 = 5\rho'^2(1-\rho'^{0\cdot4}) = 5\chi^{-4}(1-\chi^{-0\cdot8}) \tag{23}$$

when $\chi$ has non-dimensional significance (§ 9). Hence, plotting $\rho'^2q'^2$ against $\chi^2$, we can derive the form of $F$ appropriate to our assumed starting conditions. Figure 3 shows $\rho'q'$ and $q'$ $(=\chi^2\rho'q')$ plotted against $\chi$.

We deduce from (23) that $\rho'q'$, for the assumed starting conditions, cannot have a value in excess of $(5/6)^3 = 0{\cdot}578{,}704$. Consequently $M$, in (11), cannot exceed $0{\cdot}579\rho_0 a_0 T$; i.e. $\mu$, in (14) and (18), $\not> 0{\cdot}579$ $(\mu^2 \not> 0{\cdot}335)$. This accords with the conclusions of Osborne Reynolds (1886).

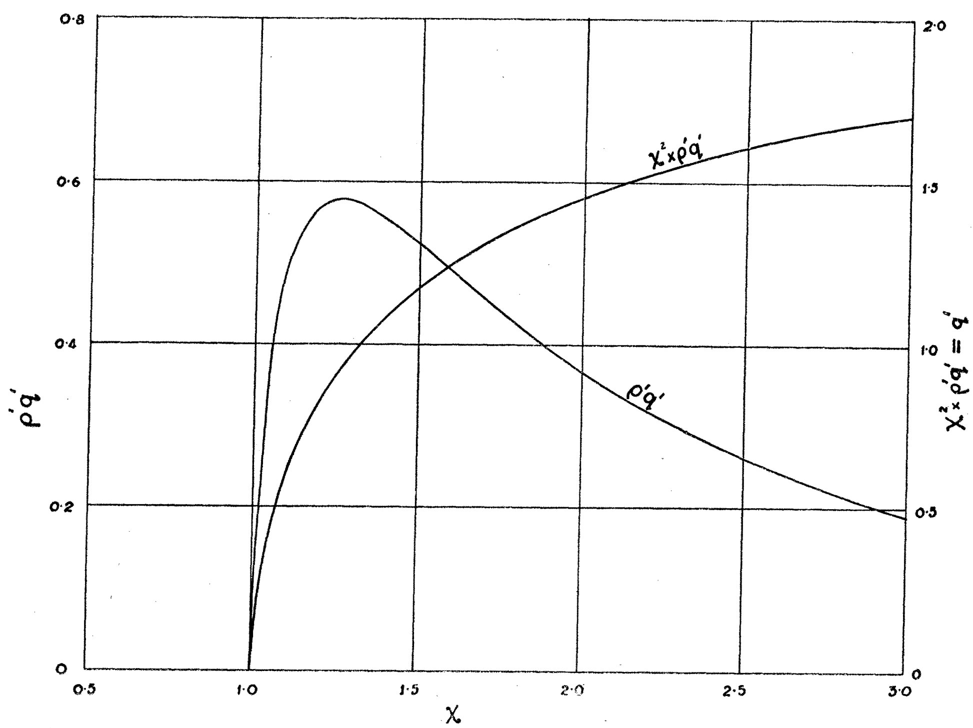


FIGURE 3

*An approximate alternative to the governing equation* (19)

11. A simplified treatment will have value in the early stages of computation. It is clear that $\psi$ will vary much faster with $\beta$ than with $\alpha$, and $\chi$, if not faster, at least not much more slowly. This means that the $\beta$-differentials will predominate in (19), so that without serious error we may suppress the $\alpha$-differentials to obtain

$$\frac{\partial}{\partial\beta}\left(\chi^2\frac{\partial\psi}{\partial\beta}\right) = 0, \quad \text{whence} \quad \chi^2\frac{\partial\psi}{\partial\beta} = \text{const.} = A \text{ (say)}. \tag{24}$$

Equally $\partial\psi/\partial\beta$ will predominate over $\partial\psi/\partial\alpha$ in the expression (22) for $\rho'^2q'^2$, so that without serious error* we may suppress the $\alpha$-differential to obtain

$$\mu h \frac{\partial\psi}{\partial\beta} = \rho'q'. \tag{25}$$

* Here, in fact, the error is of second order when $(\partial\psi/\partial\alpha)/(\partial\psi/\partial\beta)$ is small (cf. § 22).


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Now eliminating $\partial\psi/\partial\beta$, we have *approximately*

$$q' = \rho' q' \chi^2 = A\mu h, \tag{26}$$

where, by the second of (24), $\psi = \int_0^\beta \frac{\partial \psi}{\partial \beta'} d\beta' = A \int_0^\beta \chi^{-2} d\beta'.$ (27)

Applied to (27), the boundary condition (20) gives

$$1 = A \int_0^{\beta_1} \chi^{-2} d\beta, \quad \text{i.e.} \quad \mu = \mu A \int_0^{\beta_1} \chi^{-2} d\beta. \tag{28}$$

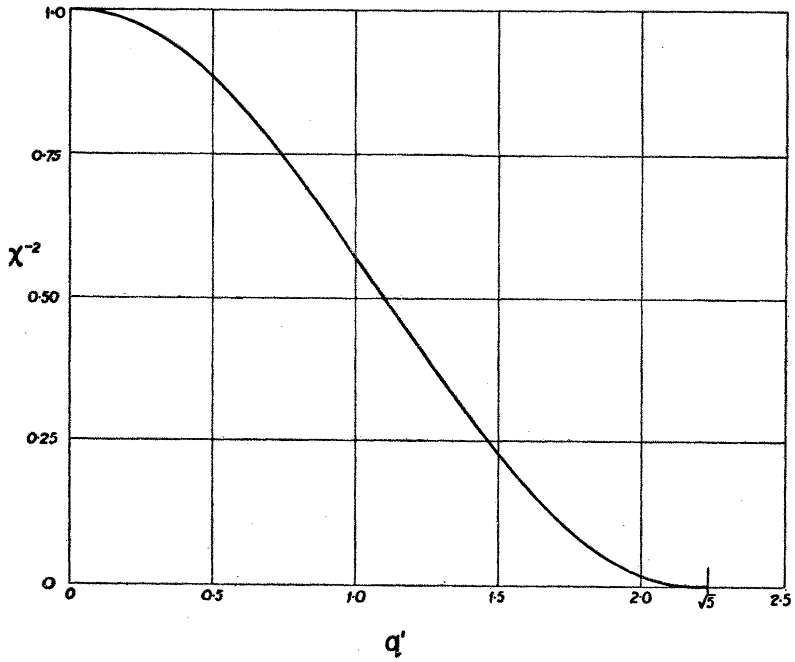


FIGURE 4*a*

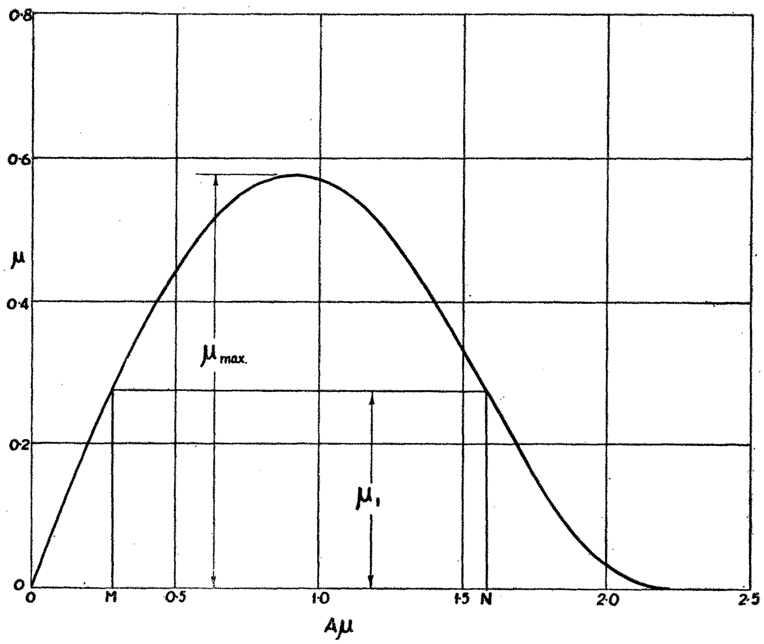


FIGURE 4*b*

$\chi^{-2}$, in these equations, is uniquely related with $q'$. Figure 3, for example, yields the curve of figure 4*a*, in which

$$q' = \rho' q' \chi^2 = \sqrt{\{5(1-\chi^{-0 \cdot 8})\}}$$

according to (23), § 10. In every case $q'$ will have a limited range ($0 < q' < \sqrt{5}$ for this last expression).

12. For every $\alpha$-line of our rectangular net we shall have (when the conformal transformation of §§ 6–8 has been effected) a known distribution of the quantity $h$. Consequently we can, for any assumed value of $A\mu$ such that $A\mu h$ nowhere exceeds the limiting value of $q'$ ($\sqrt{5}$ in figure 4*a*), deduce a distribution of $q'$ and hence, using figure 4*a* or its analytical expression, a distribution of $\chi^{-2}$. Then, evaluating the integral in the second of (28) and inserting the assumed value of $A\mu$, we have a corresponding value for $\mu$, so can construct a curve of the type shown in figure 4*b*. $\mu$ will be zero at both ends of the range of $A\mu$, and it will have a maximum value ($\mu_{\text{max.}}$, say) at some intermediate point.

Having this curve, for any value $\mu_1$ (say) less than $\mu_{\text{max.}}$ we can deduce two values of $A\mu$, and hence of $A$, whereby (28) can be satisfied: thus in figure 4*b* (on its scale of $A\mu$)

$$OM = \mu_1 A_{\text{sub.}}, \quad ON = \mu_1 A_{\text{sup.}},$$

$OM$ entailing subsonic and $ON$ supersonic velocities at the section considered. Inserting either value of $A$ in (26) and (27), we can compute $\psi$ for all nodal points in the section.

## II. Theoretical Basis of the Relaxation Approach

### *Finite-difference approximations to the governing equations*

13. Approximate treatment on the lines of §§ 11–12 entails no more than numerical integration with a use of Simpson's rule or some other approximate formula. We revert now to the exact equations of § 9.

It was shown in Part III of this series (Christopherson & Southwell 1938, § 8) that for nets of square mesh, such as are obtained by conformal transformation in the manner of §§ 6–7, the identity

$$\Sigma_{a,4}(w) - 4w_0 = a^2(\nabla^2 w)_0 \tag{29}$$

holds with neglect of terms of order $a^4$, $a^6$, ..., etc. on the right-hand side. Consequently with neglect of terms of order $a^4$ (at least) in comparison with unity we may replace (19 A), § 9, by

$$\Sigma_{a,4}(\chi\psi) - 4(\chi\psi)_0 = \psi_0\{\Sigma_{a,4}(\chi) - 4\chi_0\},$$

i.e. by

$$\Sigma_{a,4}(\chi\psi) = \psi_0 \Sigma_{a,4}(\chi). \tag{30}$$

In (29) and (30), $a$ stands for the side of each square mesh of the net, $\Sigma_{a,4}(w)$ for the sum of the $w$-values at the four points which surround 0 symmetrically at a distance $a$ (measured in terms of $\alpha$ and $\beta$).

It was shown in Part V (Gandy & Southwell 1940, § 13) that with similar approximation we may in (18 A) replace

$$\left(\frac{\partial\psi}{\partial\alpha}\right)^2 + \left(\frac{\partial\psi}{\partial\beta}\right)^2 \quad \text{by} \quad \frac{2}{a^2}\left[\psi_0^2 + \tfrac{1}{4}\{\Sigma_{a,4}(\psi^2) - 2\psi_0\Sigma_{a,4}(\psi)\}\right] = \frac{1}{2a^2}\sum_4[(\Delta\psi)^2], \text{ say,} \tag{31}$$

$\Delta\psi$ typifying the difference between $\psi_0$ and the $\psi$-value at an adjacent node, and $\Sigma_4$ denoting a summation of the four differences of this kind for the point considered. A like approximation can be used to compute $h^2$ according to (16).

*The 'relaxation pattern'*

14. The relaxation method (cf. Part III, § 11) uses these finite-difference approximations but introduces 'residual forces' to express the error of the computed displacements, and derives a 'relaxation pattern' whereby those forces can be 'liquidated' (i.e. reduced to negligible magnitudes). Here, according to (30), the residual force at 0 is defined by

$$\mathbf{F}_0 = \Sigma_{a,4}(\chi\psi) - \psi_0 \Sigma_{a,4}(\chi), \tag{32}$$

*$\chi$ and $\psi$ being now 'non-dimensional'* (§ 9); and we deduce that an infinitesimal increment $\delta\psi_0$ given to $\psi_0$ ($\psi$ being left unaltered at every other point) will entail increments to the residual forces as under:

$$\left.\begin{aligned} \delta\mathbf{F}_0 &= \Sigma_{a,4}(\psi\delta\chi) - \psi_0\Sigma_{a,4}(\delta\chi) - \delta\psi_0 . \Sigma_{a,4}(\chi), \\ \delta(\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4) &= \delta\psi_0 . \chi_0 + \psi_0 . \delta\chi_0 - (\psi_1, \psi_2, \psi_3, \psi_4)\,\delta\chi_0, \\ \delta(\mathbf{F}_A, \mathbf{F}_B, \mathbf{F}_C, \mathbf{F}_D) &= (\psi_1 - \psi_A)\,\delta\chi_1, (\psi_2 - \psi_B)\,\delta\chi_2, (\psi_3 - \psi_C)\,\delta\chi_3, (\psi_4 - \psi_D)\,\delta\chi_4. \end{aligned}\right\} \tag{i}$$

In these expressions, $\delta\chi$ denotes the increment to $\chi$ which is entailed by $\delta\psi_0$, and the suffixes 1, 2, 3, 4, $A$, $B$, $C$, $D$ relate to points so designated in figure 5. Now (18 A) and (22), § 9, show that $\chi$ is a known function of

$$\left.\begin{aligned} \rho'^2 q'^2 = \mu^2 h^2 \left\{\left(\frac{\partial\psi}{\partial\alpha}\right)^2 + \left(\frac{\partial\psi}{\partial\beta}\right)^2\right\} = \mu^2 h^2 x^2, \text{ say}, \end{aligned}\right\} \tag{ii}$$

so we may write $\qquad \chi = f(\mu^2 h^2 x^2),$

$f$ being known; and then for infinitesimal increments

$$\delta\chi = f'(\mu^2 h^2 x^2)\,\mu^2 h^2 \delta x^2,$$

where, according to (31), for an increment $\delta\psi_0$ in $\psi_0$ *only*,

$$\left.\begin{aligned} (\delta x^2)_0 &= \frac{\delta\psi_0}{a^2}[4\psi_0 - \Sigma_{a,4}(\psi)], \\ (\delta x^2)_{1,2,3,4} &= \frac{\delta\psi_0}{a^2}(\psi_0 - \psi_{1,2,3,4}). \end{aligned}\right\} \tag{iii}$$

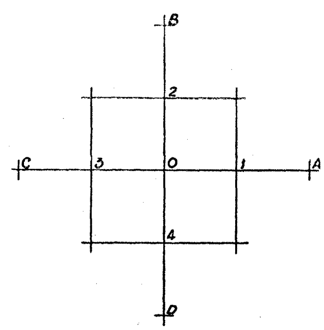


FIGURE 5

Consequently in the expressions (i) we may write

$$\left.\begin{aligned} \delta\chi_0 &= \frac{\mu^2 h^2}{a^2} f'(\mu^2 h^2 x^2)\,[4\psi_0 - \Sigma_{a,4}(\psi)]\,\delta\psi_0, \\ \delta\chi_{1,2,3,4} &= \frac{\mu^2 h^2}{a^2} f'(\mu^2 h^2 x^2)\,(\psi_0 - \psi_{1,2,3,4})\,\delta\psi_0, \end{aligned}\right\} \tag{33}$$

and then, if $\qquad G$ stands for $\dfrac{\mu^2 h^2}{a^2} f'(\mu^2 h^2 x^2) = \dfrac{\mu^2 h^2}{a^2}\dfrac{d\chi}{d(\rho'^2 q'^2)}$, by (ii), $\qquad$ (34)

it follows that when quantities of the second and higher orders in $\delta\psi_0$ are neglected

$$\left.\begin{aligned} \delta\mathbf{F}_0 &= -\delta\psi_0\{\Sigma_{a,4}(\chi) + G\sum_4 [\psi-\psi_0)^2]\},\\ \delta\mathbf{F}_1 &= \delta\psi_0[\chi_0 + G(\psi_1-\psi_0)\{\Sigma_{a,4}(\psi)-4\psi_0\}],\\ &\ldots, \text{etc.}\\ \delta\mathbf{F}_A &= -\delta\psi_0 G(\psi_1-\psi_0)(\psi_1-\psi_A),\\ &\ldots, \text{etc.}\end{aligned}\right\} \tag{35}$$

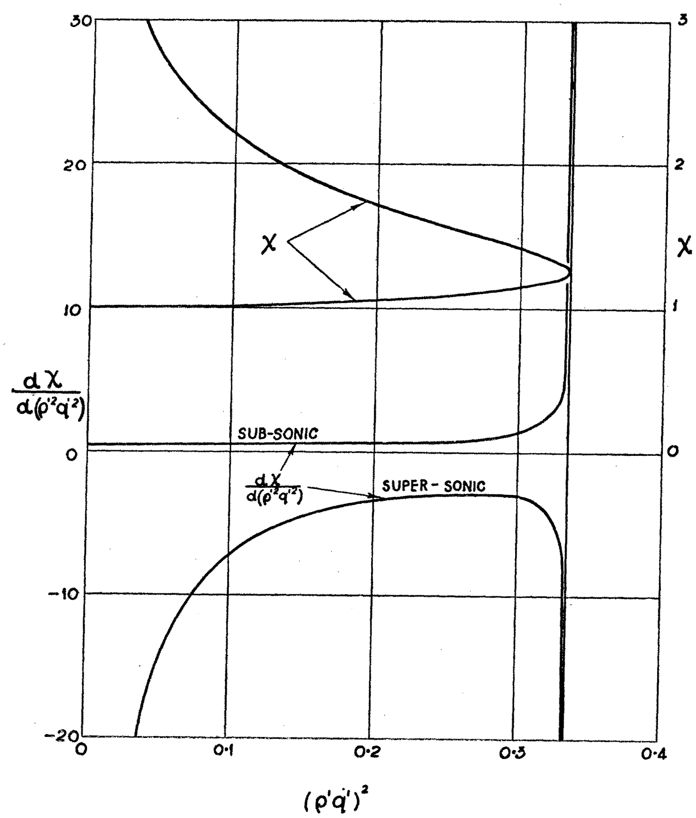


FIGURE 6

Figure 6 (deduced from figure 3) shows $\chi$ and $\dfrac{d\chi}{d(\rho'^2q'^2)}$ plotted against $\rho'^2q'^2$ as related with $\psi$ by (22). Using (35), we can for any point 0 (given $\psi$-values for 0, 1, 2, 3, 4, $A$, $B$, $C$, $D$ and $\chi$-values for 0, 1, 2, 3, 4) compute a relaxation pattern giving the effects of a *small* increment $\delta\psi_0$ to the value of $\psi_0$ upon the residual forces at all points in figure 5. Because in any event this will not hold exactly in respect of any but infinitesimal increments, and for that reason will alter as liquidation becomes more complete, there is no point in computing the *pattern* with great accuracy. What is more important is that the *residual forces* should be known with certainty, and to this end $\chi$-values must be accurately computed* in the later stages of the work.

* I.e. with all the accuracy which is permitted by the formula (31) and by the similar approximation to $h^2$.

## III. Outline of Computations effected by the Relaxation Technique

15. Given a shape of nozzle, the first requirements are (§§ 6–8) a conformal transformation of this shape into a rectangle and an evaluation of $h$ at nodal points of the resulting (square) net. Our work presumed a nozzle having the form shown in figure 1, this being the shape employed by M. W. Woods in some recent experiments (directed by Mr A. M. Binnie) on the flow of steam.* Transformation was effected with the aid of a *triangular* net,† and the resulting values of $h$ (figure 7$c$) are believed correct to at least 3 significant figures.‡ They depend, of course, on the value assumed for $\beta$ at the nozzle wall: we gave $\beta$ the value $10^5$ (cf. § 7), and for this value $h$ was found to have a maximum value 103,800 at a point on the nozzle wall close to the throat. The range of $\alpha$ was from 0 to $8{\cdot}6 \times 10^5$, so the $\alpha$-$\beta$ net (on which all subsequent operations were performed) was a rectangle having sides in the ratio 8·6 : 1 (figure 2$b$). Its mesh size $a$ was made $2 \times 10^4$ (§ 7).

16. Next, the physical properties of the fluid must be postulated, and its state at starting (from rest in a reservoir): our assumptions have been stated in § 10. Then, to define the problem completely, a value must be attached to the 'mass-flow parameter' $\mu$, defined in (14).

It has been shown (§ 10) that $\mu$ is certainly less than $0{\cdot}578_5$ ($\mu^2 < 0{\cdot}335$), this being the value at which, according to Osborne Reynolds's theory, the velocity of sound is attained at the throat. We decided to approach this value from below, and to explore the flow for a series of subsonic conditions. Calculations were accordingly made for four cases as under:

| Case | $\mu$ | $(M/T)^2$ (lb.-ft.-sec. units) |
|---|---|---|
| 1 | $0{\cdot}486{,}4_5$ | 80,000 |
| 2 | $0{\cdot}515{,}9_5$ | 90,000 |
| 3 | $0{\cdot}543{,}8_5$ | 100,000 |
| 4 | $0{\cdot}575{,}5_5$ | 112,000 |

Case 4 was expected (and was found) to give velocities just attaining to the speed of sound at a point in the throat section and on the nozzle wall.

17. Our first attack on these cases was made on the basis of the 'point-relaxation method' of §§ 13–14,—the approximate treatment of §§ 11–12 had not then (1938–40) been devised.

Starting from some assumed distribution of $\psi$ (say, $\psi = \beta$, simply), the procedure was to compute $\rho'^2 q'^2$ from (22) for mesh-points of the rectangular net, taking values of $h$ from figure 7$c$; then to deduce $\chi$-values in accordance with figure 6 and initial values of residual forces according to (32). Next, with 'patterns' computed (roughly) from (35), a partial liquidation of residuals was effected; after which the whole procedure could be repeated with its starting assumption thus improved. Although laborious the method entailed no serious difficulty, and 3-figure accuracy (at least) is claimed for the results presented in figures 8–11.

* Cf. Binnie & Woods (1938).

† Of side $T/3{\cdot}548$. Figures 7 reproduce only about one-half of the length of our actual diagrams.

‡ Full use was made of the fact that log $h$ is plane-harmonic (cf. Part V, § 17), and in fact the values given in figure 7$c$ satisfy the requirement with 5-figure accuracy.

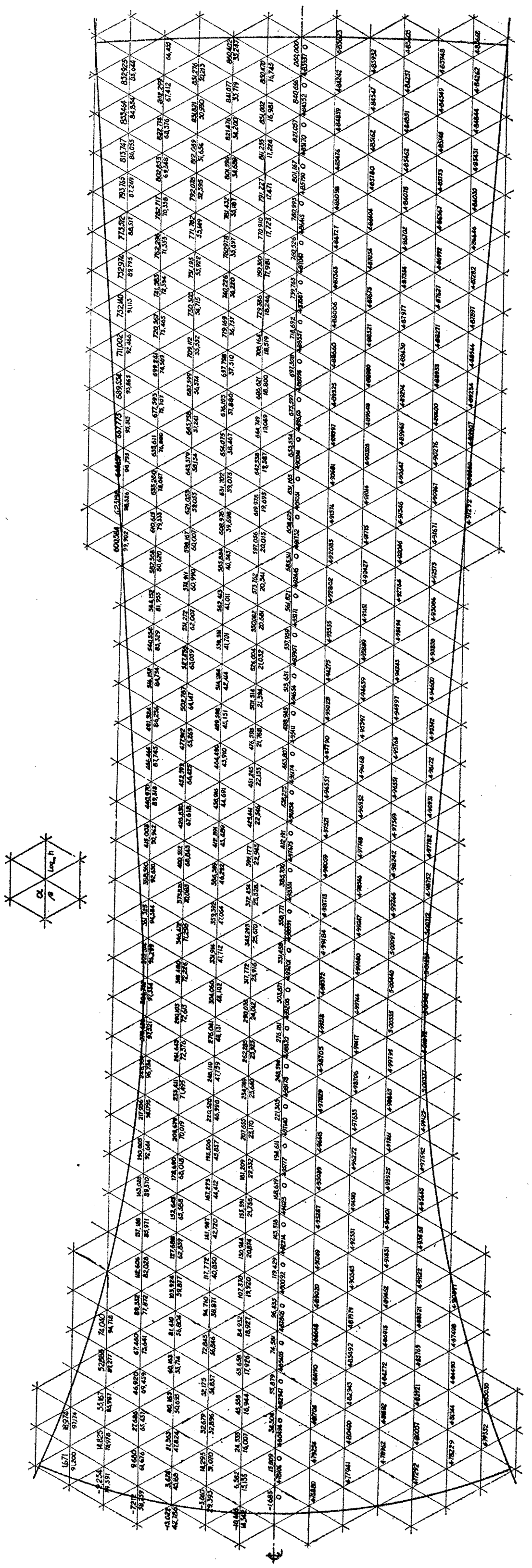

FIGURE 7a

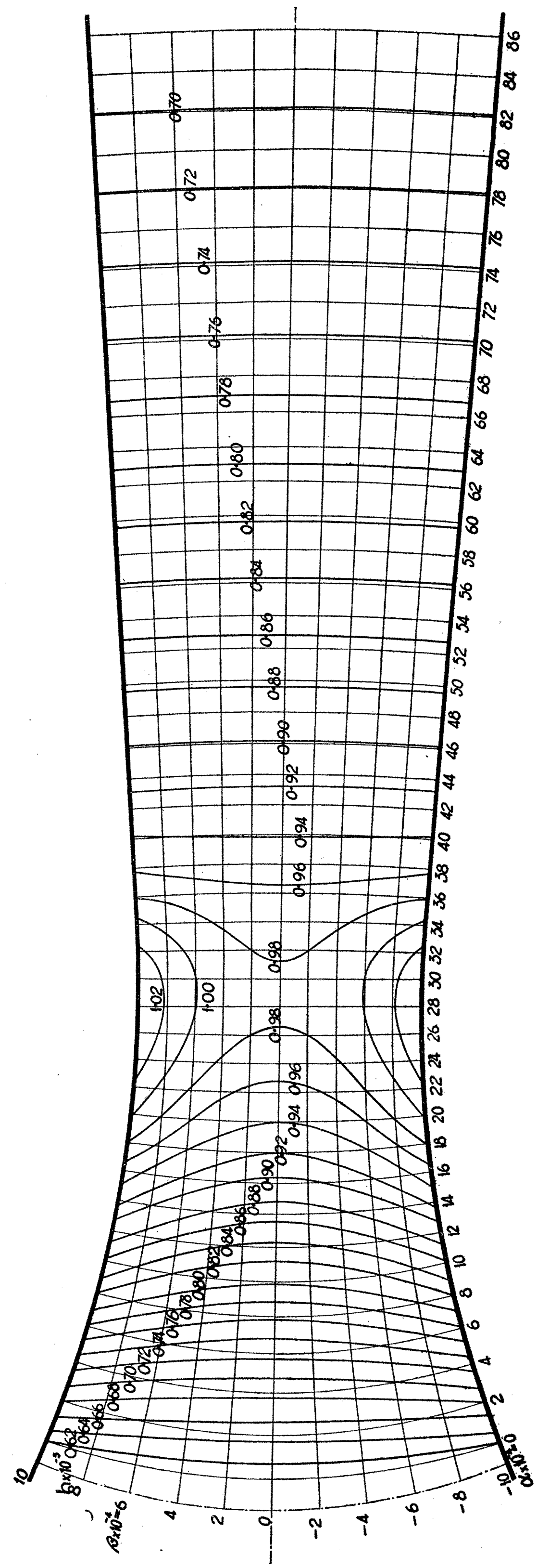


FIGURE 7b

18. Under supersonic conditions, on the other hand, this technique failed to yield definite results. The failure was not unexpected, in view of similar failure of the electrical tank: Taylor & Sharman (1928) found that below the speed of sound they could satisfy (4) together with the boundary condition by a process entailing successive modifications of an assumed density distribution, but above the speed of sound these modifications become oscillatory or divergent, and in consequence the method failed. Our own difficulty in treating flow through a nozzle we ascribed to some kind of instability appearing downstream of the throat, whereby any disturbance (implicit in an assumed solution) would be magnified without limit. (Figure 6 shows that $G$, in (34), tends to an infinite value corresponding with the speed of sound; so near the throat, and for supersonic flow, the 'relaxation patterns' which come from (35) must involve very large numbers.)

19. It was in these circumstances that the alternative method of §§ 11–12 was devised. In that method, for each $\alpha$-line of the rectangular net a curve (typified by figure 4*b*) is derived from which, for any permissible value of $\mu$, two acceptable values of $A$, in (26) and (27), can be found. One $A$-value yields a subsonic, the other a supersonic system of velocities.

It was shown (§ 12) that in the problem of this paper (§ 10), since $q'$ has an analytical expression in terms of $\chi$, (26) can be replaced by

$$\mu h A = \sqrt{\{5(1-\chi^{-0\cdot 8})\}}, \quad \text{giving} \quad \chi^{-2} = \left(1 - \frac{\mu^2 h^2 A^2}{5}\right)^{2\cdot 5}.$$

Consequently (28) can be written in the form

$$1 = A\int_0^{\beta_1} \chi^{-2} d\beta \text{ (as before)} = A\int_0^{\beta_1}\left(1-\frac{\mu^2h^2A^2}{5}\right)^{2\cdot 5} d\beta, \tag{36}$$

and a similar substitution can be made for (27).

### *Extension of the alternative (approximate) method of* §§ 11–12

20. Before conclusions could be based on the method, the order of its accuracy had to be decided. To this end we applied it to two of the four cases, listed in § 16, which had already been treated by

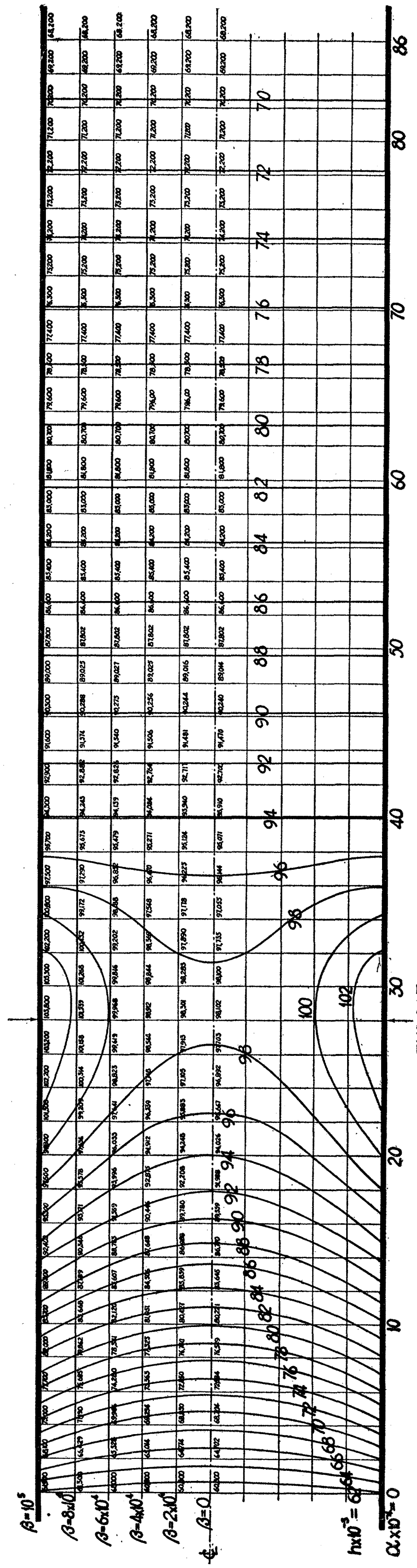

Figure 7*c*

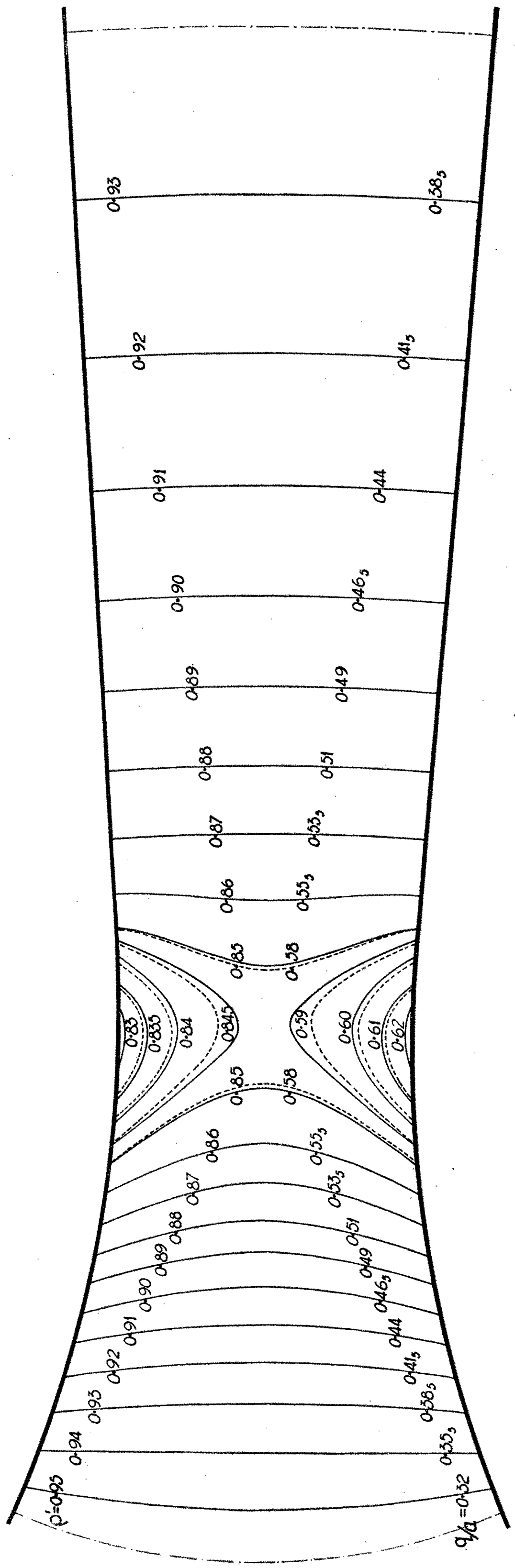

FIGURE 8. $m^2 = 80,000$.

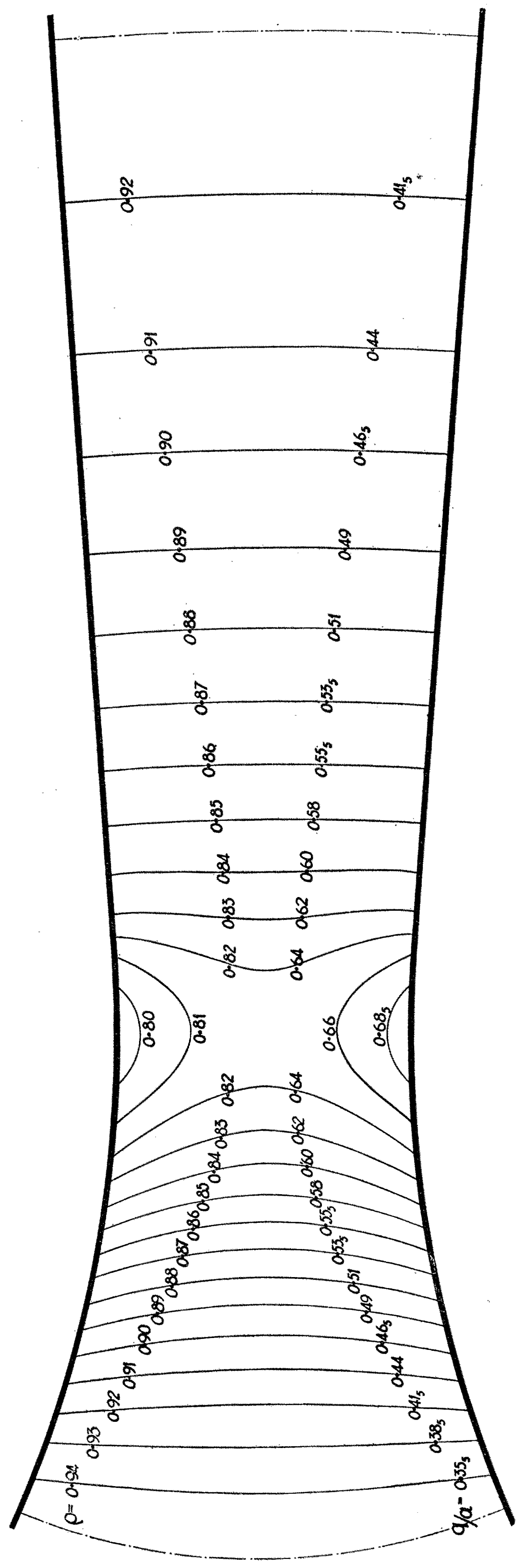

FIGURE 9. $m^2 = 90,000$.



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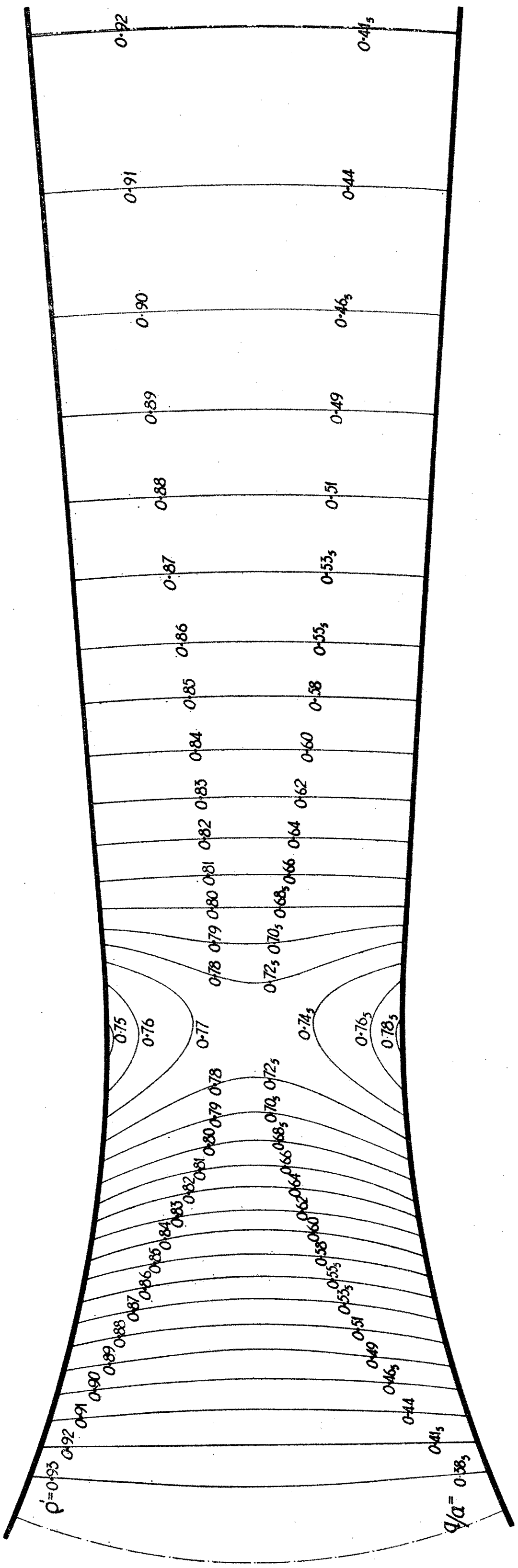


FIGURE 10. $m^2 = 100{,}000$.

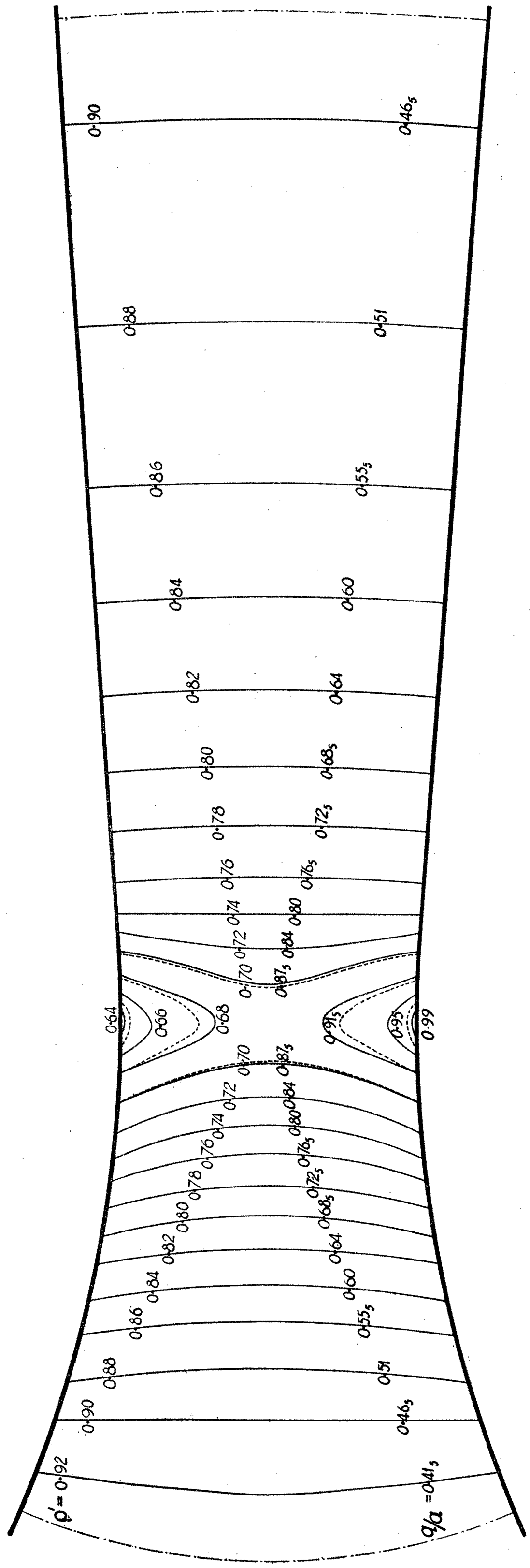


FIGURE 11. $m^2 = 112{,}000$.

point-relaxation' methods; our criterion of its accuracy being the approximation to which it reproduced the previously computed values of $\rho'$ $(=\chi^{-2})$. Grateful acknowledgment is made of assistance given by Dr L. Fox and Miss G. Vaisey in this part of the investigation.

Broken lines in figures 8 and 11 are contours deduced by the alternative method: only those shown were distinguishable from the full-line contours (relating to § 17). The smaller of each pair of values of $A$ (§ 19) led to these subsonic solutions.

21. These results being deemed satisfactory, we proceeded to employ the method in the supersonic range and, for the resulting $\psi$-values, to deduce residual forces according to (32). These were negligible for sections downstream from the throat by more than the whole throat width ($2T$ in figure 1). They were however, near the throat, too large to be accepted; and previous experience (§ 18) gave little reason to expect that 'point relaxation' would serve to liquidate them. Accordingly we sought to improve the accuracy of the alternative method by an extension on *iterative* lines.

22. The first of (24) was an approximation to

$$\frac{\partial}{\partial\beta}\left(\chi^2\frac{\partial\psi}{\partial\beta}\right) = -\frac{\partial}{\partial\alpha}\left(\chi^2\frac{\partial\psi}{\partial\alpha}\right), \tag{37}$$

which can be identified with (19 A), § 9. Giving the term on the right its value as deduced in the manner of § 21, we can integrate (37) to obtain a second approximation to $\chi^2(\partial\psi/\partial\beta)$.

(25), similarly, was an approximation to (22); but in this instance, $\partial\psi/\partial\alpha$ being small in relation to $\partial\psi/\partial\beta$, the error is of the second order, so can be accepted. Accordingly we now combine

$$\mu h\frac{\partial\psi}{\partial\beta} = \rho' q' \qquad (25)\ bis$$

with

$$\chi^2\frac{\partial\psi}{\partial\beta} = A - \int\frac{\partial}{\partial\alpha}\left(\chi^2\frac{\partial\psi}{\partial\alpha}\right)d\beta, \tag{38}$$

which comes from (37) and replaces (24). Thereby we obtain, in place of (26)

$$q' = \rho' q'\chi^2 = A\mu h - \mu h\, F(\beta), \tag{39}$$

and in place of (28)

$$1 = A\int_0^{\beta_1}\chi^{-2}d\beta - \int_0^{\beta_1}\chi^{-2}F(\beta)\,d\beta, \tag{40}$$

where $A$ as before is a constant of integration, and

$$F(\beta) = \int_0^{\beta}\frac{\partial}{\partial\alpha}\left(\chi^2\frac{\partial\psi}{\partial\alpha}\right)d\beta' \tag{41}$$

is a function of $\beta$ which, as stated above, we can with sufficient approximation deduce from our first solution.

23. When (39) and (40) are thus substituted for (26) and (28), the procedure of § 12 ceases to be applicable for the reason that $q'$ is *not* deducible from (39), as it was from (26), for an assumed value of $A\mu$. But it is the basis of our method that $F(\beta)$ will be small and that the new value of $\mu$ will differ but little from $\mu_0$, the value which corresponds with the assumed

$A\mu$ according to our first approximation: consequently it will be sufficiently accurate to replace (39) and (40) by

$$q' = \rho' q' \chi^2 = A\mu h - \mu_0 h F(\beta),$$

and by

$$\left.\begin{aligned} \mu = A\mu \int_0^{\beta_1} \chi^{-2} d\beta - \mu_0 \int_0^{\beta_1} \chi^{-2} F(\beta)\, d\beta, \end{aligned}\right\} \quad (42)$$

and then ($\mu_0$ and $F(\beta)$ being known) the construction of a curve of the type of figure 4*b* can proceed as before.

Any approximation, taken as a starting solution, can be thus improved. So our iterative process seems capable of giving any required accuracy within the subsonic range.

### *Application of the extended method to supersonic conditions*

24. But a new difficulty is confronted when the *supersonic* regime is in question, because $\mu$ must then have its limiting value,—namely, that for which Osborne Reynolds's simplified theory yields the approximation 0·579 (§10),—and this value is *not* known in advance. If we assume too low a value, then the regime will be subsonic throughout; whereas too high a value is physically inadmissible and can lead to no real results. It is of great importance therefore—even in an approximate treatment—to have the closest possible estimate of the correct (i.e. limiting) value.

Fortunately this presents no great difficulty. In the simplified treatment of §§11–12 (and in the extension outlined in §23) we construct for every section a curve, of the type of figure 4*b*, in which $\mu$ ranges from 0 to some maximum value $\mu_{\text{max.}}$. Only values less than $\mu_{\text{max.}}$ are admissible, and so (since in any possible regime $\mu$ will have the same value for every section) the wanted (limiting) value of $\mu$ is the least $\mu_{\text{max.}}$ which is discoverable by our analysis. It will be the $\mu_{\text{max.}}$ for some section near the throat, but it is not certain that this section will be one of those that we have investigated. However, having values of $\mu_{\text{max.}}$ for three or four values of $\alpha$ near the throat, it is an easy matter to deduce (by customary finite-difference methods) the $\alpha$-value for which $\mu_{\text{max.}}$ has its minimum value, and thence to compute this minimum. Now accepting the estimated value ($\mu$, say), and reverting to the curves (of type figure 4*b*) which have been constructed from other sections, it is again an easy matter to deduce for each of them a value of $A_{\text{sup.}}$ (§12) and with this to compute $\psi$ for the relevant section. Proceeding in this manner, *provided that the iterative process is convergent*, we shall arrive, finally, at a *unique supersonic regime*, entailing a definite pressure at every point.

## Conclusion

25. The alternative ('strip integration') method of §§11–12 and §§22–3 has no feature properly described as 'relaxational', consequently an investigation by that method of the supersonic regime is hardly appropriate to the present series, but should form the subject of a separate paper. We had intended to complete it before communicating this partial (subsonic) study; but that decision has been altered by circumstances (cf. §1) which prevent any further collaboration.

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